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General equation for heat transfer for laminar flow in ducts of arbitrary cross-sections

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Abstract—This paper presents the general equations for heat transfer calculations for constant wall temperature in laminar developed flow in ducts of arbitrary cross sections. The results obtained from these equations compare well with the theoretically-calculated values available in the literature for circular, rectangular, triangular, elliptical and parallel plate ducts. The maximum and minimum deviations by these comparisons are −8.7 and +8.0% respectively.

INTRODUCTION

WE CAN calculate the heat transfer in ducts of arbitrary cross-sections with the definition of the equivalent diameter only in turbulent flow. In laminar flow, it is not sufficient to define an equivalent diameter, because the boundary layer of each wall is influenced by the other wall. Therefore, one needs other additional quantities to describe the heat transfer and pressure drop. This is shown by Yılmaz [1]. By using other quantities, it was possible to obtain a general equation for pressure drop in ducts of arbitrary cross-sections. There does not exist a general equation for the calculation of heat transfer for constant wall temperature for laminar developed flow in ducts of arbitrary cross-sections. In this work, such equations will be given.

HEAT TRANSFER BY DEVELOPED FLOW

It is assumed that heat is transferred by constant wall temperature. Heat transfer begins at $z = 0$, where the flow is already developed.

Heat transfer by constant wall temperature is described by the Nusselt number Nu, which is defined as follows:

$$Nu = \frac{hd_e}{k}. \qquad (1)$$

Nu is constant for hydrodynamically and thermally developed flow (HTDF) and is denoted as Nu_∞. In Table 1, Nusselt numbers for various cross-sections for hydrodynamically and thermally developed flow are given [2].

For hydrodynamically-developed and thermally developing flow Nu can be calculated with the help of the following equation given by Yılmaz and Cihan [3].

$$z^* \to 0: \quad Nu = \frac{1.615\Phi}{(z^*/\Psi)^{1/3}} \qquad (2)$$

where z^* is a dimensionless number

$$z^* = \frac{z}{d_e}\frac{1}{Re\,Pr}. \qquad (3)$$

Re and Pr are Reynolds and Prandtl numbers respectively:

$$Re = \frac{ud_e}{\nu} \qquad (4)$$

$$Pr = \frac{\nu}{a}. \qquad (5)$$

Here, u is the average velocity, ν the kinematic viscosity, d_e the equivalent diameter and a the thermal diffusivity. Ψ is the shape factor which is defined for the calculation of pressure drop for laminar developed flow:

$$\Delta P = \Psi \frac{64}{Re}\frac{L}{d_e}\frac{\rho u^2}{2}. \qquad (6)$$

Ψ can be determined for different cross-sectional areas according to the formulas given by Yılmaz [1].

$$\Psi = 1 + \frac{(3/8)d^{*2}(3-d^*)-1}{1+0.33d^{*2.25}/(n-1)}. \qquad (7)$$

ρ is the density. Φ is determined by the following equation [3]:

Table 1. Nusselt number for HTDF for various shaped cross-sections

Cross-section	Nu_∞
Circular	3.657
Parallel plate	7.541
Equilateral triangle	2.46
Square	2.976

NOMENCLATURE

- a thermal diffusivity of fluid
- b length of the cross-section of the duct
- A cross-sectional area of the duct
- d diameter
- h heat transfer coefficient
- k thermal conductivity
- L length of the duct
- m constant, equation (21)
- n number of equivalent circles, equation (10)
- Nu Nusselt number
- P periphery of the duct
- T temperature
- u velocity
- x coordinate of the cross-sectional area
- X variable defined in equation (27)
- y coordinate of the cross-sectional area, Fig. 2
- Y variable defined in equation (26)
- z axial coordinate.

Greek symbols

- Δp pressure drop
- ε percent deviation
- ϕ heat transfer factor, equation (13)
- ν kinematic viscosity
- ρ density
- Φ heat transfer factor, equation (8)
- Ψ shape factor for developed flow.

Superscript and indices

- $*$ dimensionless
- e equivalent
- max maximum
- ∞ for $n \to \infty$ or $z \to \infty$.

$$\Phi = 1 + \frac{[3(d^*/2)^{7/8}/(1+d^*)] - 1}{1 + 0.25/(n-1)}. \tag{8}$$

For arbitrary cross sections, Nu must be at least dependent on z^*, d^* and n [1]. d^*, n and d_e are defined as follows:

$$d^* = \frac{d_e}{d_{max}} \tag{9}$$

$$n = \frac{P}{P_e} = \frac{A}{A_e} \tag{10}$$

$$d_e = \frac{4A}{P}. \tag{11}$$

Here, P and P_e are the periphery of the duct and of the equivalent circle, respectively. A and A_e respectively are the cross sectional area of the duct and of the circular tube with the equivalent diameter d_e. d_{max} is the maximum diameter of the circle suiting the cross section of the actual duct. In Fig. 1, d_{max} of various ducts are presented. We can now expect the following relationship for Nu:

$$Nu = f(z^*, d^*, n). \tag{12}$$

NUSSELT NUMBER FOR HYDRODYNAMICALLY AND THERMALLY DEVELOPED FLOW

For hydrodynamically and thermally developed flow we define

$$\phi = \frac{Nu_\infty}{3.657}. \tag{13}$$

Nu is the Nusselt number for HTDF. It is seen that ϕ is the ratio of Nu_∞ of a duct to the $Nu_\infty = 3.657$ of the circular duct. For $n \to \infty$, ϕ is denoted by ϕ_∞ and $n \to \infty$ means that one dimension of the cross section is very large compared to the other. The definition of n is given in equation (10). A duct for $n \to \infty$ is shown in Fig. 2.

Maclaine-Cross [4] gives the integral equation for ϕ_∞

$$\phi_\infty = 8.248 \frac{\int_0^b y \, dx \int_0^b y^3 \, dx}{P^2 d_{max}^4}. \tag{14}$$

One can write for the cross sectional area

$$A = \int_0^b y \, dx \tag{15}$$

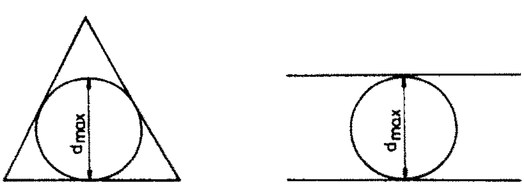

FIG. 1. Explanation of the maximum diameter d_{max}.

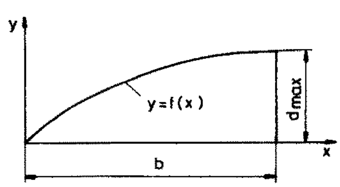

FIG. 2. Example for a duct with $n \to \infty$.

with the definitions

$$y^* = \frac{y}{d_{max}}; \quad x^* = \frac{x}{b} \tag{16}$$

using equations (9) and (11) and considering $P \approx 2b$ one gets

$$d^* = 2\int_0^1 y^* \, dx^*. \tag{17}$$

With equations (9), (16) and (17) one obtains from equation (14)

$$\phi_\infty = 1.031 d^* \int_0^1 y^{*3} \, dx^*. \tag{18}$$

We can calculate ϕ_∞, if we have a proper equation for y^* as a function of x^*. Such an equation has to fulfil only the conditions

$$x^* = 0: \quad y^* = 0$$
$$x^* = 1: \quad y^* = 1. \tag{19}$$

There are many equations fulfilling these conditions. The following simple equation suits these conditions and would be appropriate for cross section peripheries without a turning point:

$$y^* = x^{*m} \tag{20}$$

with this equation, one obtains from equation (17)

$$m = \frac{2}{d^*} - 1. \tag{21}$$

Applying equations (20) and (21) to equation (18) we get the following relationship for ϕ_∞:

$$\phi_\infty = 0.5155 \frac{d^{*2}}{(3-d^*)}. \tag{22}$$

In Table 2, theoretically obtained values Nu_∞ and ϕ_∞ for various cross sections are given [4]. These values and the derived equation (22) are presented in Fig. 3. The theoretically-obtained values are described with a maximum deviation of +6.6% as seen in Table 2.

GENERAL EQUATION FOR ϕ

For $n \neq \infty$, ϕ is dependent on d^* and n. Using the theoretically obtained values given in the literature [4] one can derive the following equation with the methods given in ref. [8].

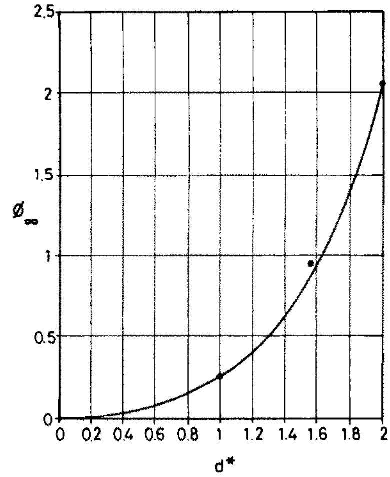

FIG. 3. Heat transfer factor ϕ_∞ for $n \to \infty$ vs dimensionless diameter d^*.

$$\phi = 1 + \frac{\phi_\infty - 1}{1 + 1/(n-1)} + \Delta\phi \tag{23}$$

$$\Delta\phi = \Delta\phi_{max} \frac{0.95(n-1)^{0.5}}{1 + 0.038(n-1)^3} \tag{24}$$

$$\Delta\phi_{max} = \frac{7 \times 10^{-3} d^{*8}}{(1 + 10 d^{*-28})(1 + 64 \times 10^{-8} d^{*28})^{0.5}}. \tag{25}$$

In Table 3, ϕ values calculated with equation (23) are compared with the theoretically obtained values given in the literature [2]. The maximum and minimum deviations between these values are less than +5.9 and −7.6, respectively, as seen from Table 3.

GENERAL EQUATION FOR Nu

To obtain general equations for Nu, we define new dimensionless quantities

$$Y = \frac{Nu}{Nu_\infty} \tag{26}$$

$$X = \frac{z^* Nu_\infty^3}{\Psi \Phi^3}. \tag{27}$$

With this coordinate transformation we obtain only one curve for $z \to 0$ and $z \to \infty$. The Nusselt number values for parallel plates, circular, triangular, rectangular and elliptical ducts are given in Fig. 4 where this is stated clearly. From these single curves

Table 2. Comparison of the results of equation (22) with theoretical data

Duct	Ref.	d^*	Nu	ϕ_∞ Ref. value	ϕ_∞ Equation (22)	ε (%)
Triangular	[5]	1	0.943	0.2578	0.2578	0
Elliptical	[6]	1.571	3.488	0.9537	0.8903	6.6
Parallel plate	[7]	2	7.541	2.062	2.062	0

Table 3. Comparison of equation (23) with the theoretical data

Cross sections	Ref.	d^*	n	ϕ Ref. value	ϕ Equation (23)	ε (%)
Triangular	[5]	1.000	5.770	0.399	0.386	−3.09
		1.000	4.339	0.440	0.429	−2.43
		1.000	3.267	0.495	0.485	−1.90
		1.000	2.564	0.546	0.547	0.34
		1.000	2.088	0.607	0.613	1.12
		1.000	2.017	0.618	0.626	1.34
		1.000	1.838	0.645	0.662	2.65
		1.000	1.784	0.654	0.674	3.10
		1.000	1.681	0.669	0.699	4.60
		1.000	1.666	0.672	0.703	4.72
		1.000	1.653	0.675	0.707	4.78
		1.000	1.666	0.669	0.703	5.19
		1.000	1.678	0.669	0.700	4.72
		1.000	1.744	0.656	0.683	4.24
		1.000	1.855	0.639	0.658	3.04
		1.000	2.856	0.519	0.518	−0.13
		1.000	4.920	0.410	0.409	−0.24
		1.000	5.251	0.401	0.399	−0.39
Rectangular	[7]	1.000	1.273	0.814	0.841	3.33
		1.166	1.309	0.841	0.865	2.90
		1.200	1.326	0.853	0.870	2.09
		1.333	1.432	0.927	0.907	−2.12
		1.500	1.697	1.081	1.043	−3.45
		1.600	1.989	1.214	1.208	−0.42
		1.714	2.599	1.405	1.408	0.24
		1.778	3.222	1.531	1.487	−2.84
Parallel plate	[7]	2.000	∞	2.062	2.062	0
Circular	[6]	1.000	1.000	1.000	1.000	0
Elliptical	[6]	1.108	1.018	1.003	0.989	−1.34
		1.297	1.188	1.023	0.945	−7.60
		1.464	1.864	1.037	0.995	−4.00
		1.535	3.394	1.018	1.077	5.87
		1.599	6.575	0.997	0.956	−4.07

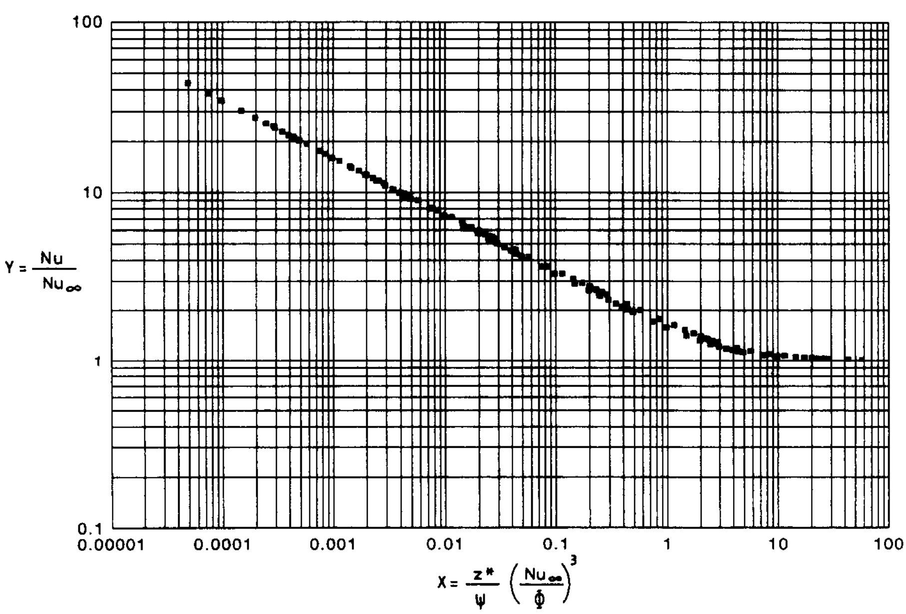

FIG. 4. Graphical representation of numerical results according to the definition in equations (26) and (27).

Table 4. Comparison of equations (30) and (31) with numerical data given in literature

Duct	Ref.	d^*	n	ε (%) equation (30)	ε (%) equation (31)
Circular	[9]	1	1	−0.47/−3.89	0.36/−4.62
Parallel plate	[9]	2	∞	0.24/3.70	0.13/4.91
Square	[10]	1	1.273	−3.52/0.58	−2.58/0.21
Equilateral triangular	[11]	1	1.653	−4.36/−1.29	−2.94/−0.93
Rectangular	[11]	1	1.273	−3.93/1.39	−2.99/0.77
		1.5	1.697	−0.86/4.16	−1.55/4.65
		1.333	1.432	−2.98/−0.03	−3.41/0.70
		1.6	1.989	0.48/3.22	3.82/0.27
		1.666	2.291	2.60/1.24	2.62/1.69
		1.714	2.599	3.32/1.59	4.20/2.38

Table 5. Comparison of equations (30) and (31) with our own numerical results

Duct	d^*	n	ε (%) equation (30)	ε (%) equation (31)
Rectangular	1	1.273	−3.33/4.17	5.62/−3.04
	1.333	1.432	−2.47/1.73	−3.36/1.71
	1.6	1.989	3.86/−2.59	4.17/−1.21
	1.818	3.851	−3.23/8.02	−2.14/7.44
Isosceles triangular	1	1.666	−8.74/2.30	−7.07/2.83
	1	1.855	−2.62/4.98	−1.97/6.15
	1	2.489	−1.61/5.68	−0.74/7.08
	1	6.492	−2.72/6.06	−3.53/7.46
Right triangular	1	1.855	−7.19/3.68	−5.52/4.32
	1	3.311	−2.26/3.42	−0.74/4.79
	1	7.052	−1.37/3.55	−2.30/5.02
Elliptical	1.107	1.018	−2.66/−1.06	−3.05/0.45
	1.230	1.100	1.40/4.70	0.52/4.54
Circular	1	1	−4.34/−2.46	−4.60/−0.93

we then get the equation with the method described in ref. [8].

$$Y = \left[1 - \frac{0.8}{X^{2/3}} + \frac{4.212}{X}\right]^{1/3} \quad (28)$$

$$Y = 1 + \frac{1.615X^{-1/3}}{(1+1.88X^{1/3}+3.93X^{4/3})^{1/2}}. \quad (29)$$

Using the definition in equations (26) and (27) we rewrite equations (28) and (29):

$$Nu = Nu_\infty\left[1 + \frac{4.212\Psi\Phi^3}{z^*Nu_\infty^3} - 0.8\left(\frac{\Psi\Phi^3}{z^*Nu_\infty^3}\right)^{2/3}\right]^{1/3} \quad (30)$$

$$Nu = Nu_\infty + \frac{1.615\Phi/(z^*/\Psi)^{1/3}}{\left[1+1.88\left(\frac{z^*Nu_\infty^3}{\Psi\Phi^3}\right)^{1/3}+3.93\left(\frac{z^*Nu_\infty^3}{\Psi\Phi^3}\right)^{4/3}\right]^{1/2}}. \quad (31)$$

These equations are compared with the theoretically obtained values for ducts with circular, parallel plates, elliptical and rectangular cross-sectional areas in Tables 4 and 5. The maximum and minimum deviations are −8.7 and +8.0% respectively.

CONCLUSIONS

Equations (30) and (31) have been obtained for the calculation of heat transfer by constant wall temperature in laminar developed flow for ducts with arbitrary cross-sectional shapes. It is shown that the given equations compare well with the theoretically-obtained values for circular, parallel plates, rectangular, isosceles triangular, right triangular and elliptical ducts. The maximum and minimum deviations by these comparisons are −8.7 and +8.0% respectively. It is expected that these equations will describe Nusselt numbers for other shaped ducts with sufficient accuracy.

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